

## On-line Scheduling in Assembly Processes

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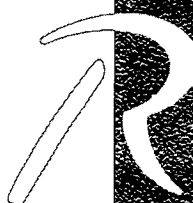
# *On-line Scheduling in Assembly Processes*

Fabrice Chauvet - Jean-Marie Proth

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# On-line Scheduling in Assembly Processes

Fabrice CHAUVET\* and Jean-Marie PROTH\* and \*\*

## ABSTRACT

The assembly system under consideration is composed with several machines, and some of these machines may be identical or able to perform the same operations. The manufacturing system is fully automated and, semi-finished products or components are not stored during the process. A limited flexibility exist since the manufacturing times can be extended within certain limits at the expense of the unavailability of the resource. There are no conflicts between the resources; in other words, the same machine cannot be used to perform different operations for a same product. Due to the intensity of the flow of products to be manufactured, it is not allowed to reschedule products which have been previously scheduled. Thus, when a new product requirement arrives in the system, we have to take advantage of the idle time windows. The goal is to complete the product as soon as possible. We give a real-time scheduling algorithm which guarantees an optimal makespan to any product which arrives in the assembly system. A numerical example is provided to illustrate this approach.

## KEYWORDS

Makespan optimization, Real-time scheduling, On-line scheduling, Assembly system.

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# Processus d'assemblage en temps réel

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## RESUME

Le système d'assemblage considéré est composé de plusieurs machines. Certaines de ces machines peuvent être identiques ou différentes mais en mesure d'exécuter les mêmes opérations. Le système est automatisé, et il n'est pas possible de stocker des composants ou des produits semi-finis au cours de la fabrication. Une flexibilité limitée du système existe : il est possible d'étendre les temps opératoires dans certaines limites, mais dans ce cas la ressource concernée n'est pas disponible pour exécuter une autre opération. Il n'y a pas de conflit entre les ressources ; en d'autres termes, une même machine n'est jamais utilisée pour exécuter plus d'une opération sur le même produit. Du fait de la fréquence d'arrivée des produits à fabriquer, il n'est pas possible de remettre en question l'ordonnancement des produits déjà ordonnancés. Par conséquent, lorsqu'une nouvelle commande arrive dans le système, il faut tirer avantage au mieux des périodes de vacuité des ressources. L'objectif est d'exécuter la commande au plus tôt. Nous proposons un algorithme qui garantit l'obtention d'un temps de fin de fabrication (makespan) aussi court que possible. Cet algorithme est illustré par un exemple numérique.

## MOTS CLES

Ordonnancement temps réel, makespan, système d'assemblage.

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## 1. INTRODUCTION

The assembly system considered in this paper is fully automated. Customers' requirements stream in the system, and each new requirement should be scheduled as soon as it enters the system, the goal being to reduce its completion time as much as possible. In other words, the goal is to minimize the makespan of each product considered individually. Due to the importance of the production flow, it is impossible to reschedule a product previously scheduled. In other words, when a customer's requirement arrives in the system, and this can happen at any time, the goal is to take advantage of the idle periods available on the resource to complete the product as soon as possible.

Two types of operation have to be performed, that is the processing operations, which transform the state of components, and the assembly operations, which assemble two or more components, in order to obtain another component or the finished product.

We assume that a component is not allowed to wait in front of a resource: the next operation to be performed on the component should start as soon as the current ends. This constraint is very strong, mainly since several component which have to be assembled should be completed exactly at the same time. This constraint aims at reducing the work-in-process (WIP) as much as possible.

Nevertheless, limited flexibility exists: the processing time associated with a given operation may be extended within certain limits, but the resource on which an operation is performed is unavailable for another operation as long as the current operation is not completed. In particular, we can find this kind of limited flexibility in chemical processes.

Numerous papers have been dedicated to no-wait or blocking problems. In these problems, it is assumed that no intermediate buffers exist between the resources.

Callahan [1971] uses a queuing model to study a no-wait processing in the steel industry. Chu and all [to appear], and Chauvet and all [1997] consider the surface treatment which is a no-wait problem. McCormick and all [1989] study a cyclic flowshop with buffers which can be transformed into a blocking problem by considering buffers as resources with totally flexible processing times, i.e. processing times which can take any value between 0 and  $+\infty$ . Hall and Sriskandarajah [1996] make a survey on scheduling problems with blocking and no-wait, while Rachamadugu and Stecke [1994] classify the FMS scheduling procedures. This paper should be considered as an intermediate approach: some flexibility exists in the model, but this flexibility can be limited.

## 2. NOTATIONS

Each operation of the assembly process is defined by its minimal and maximal processing times. For instance, operation  $i$  is characterized by  $\theta_i$ , minimal processing time, and  $\theta_i + \delta_i$ , maximal processing time. Two kinds of operations can be found in an assembly process, that is:

(i) The processing operations, which transform the state of a component or a semi-finished product. A processing operation has at most one predecessor and one successor.

(ii) The assembly operations, which assemble several components. The last operations performed on these components are the predecessors of the assembly operation. In an assembly process, an assembly operation has at most one successor. It has no successor when the result of the assembly operation is the finished product.

Each operation  $i$  is characterized by the set of its predecessors denoted by  $\Gamma_i^-$  and the set of its successor denoted by  $\Gamma_i^+$ . Note that, in an assembly process,  $\Gamma_i^-$  may contain several elements or may be empty.  $\Gamma_i^+$  contains at most one element.

Thus, an assembly process  $A$  is characterized by the list  $\{1, 2, \dots, n\}$  of its operations, each operation  $i \in \{1, 2, \dots, n\}$  being characterized by four elements:  $\{\theta_i, \delta_i, \Gamma_i^-, \Gamma_i^+\}$ . We denote by  $v_i$  the starting time of operation  $i$ ,  $i \in \{1, 2, \dots, n\}$ . Since operation  $j$  where  $j \in \Gamma_i^+$  and  $\Gamma_i^+ \neq \emptyset$ , should start as soon as the current operation  $i$  ends,  $v_j$  is the completion time of operation  $i$ .

### 3. PROBLEM FORMULATION

The manufacturing system is composed with several machines, and some of these machines may be identical or able to perform the same operations. Since other products have been scheduled previously, the machines are partially busy. The manufacturing system being fully automated, semi-finished products or components are not stored during the process. The only flexibility available in the system is given by the  $\delta_i$ ' values, which express the fact that manufacturing times can be extended. This is in some way equivalent to the storage of semi-finished products or components, but makes the machines unavailable for other operations.

For each operation  $i$ , we have a series of available windows denoted by:  $[\alpha_1^i, \beta_1^i]$ ,  $[\alpha_2^i, \beta_2^i]$ , ...,  $[\alpha_{q_i}^i, \beta_{q_i}^i]$ . These windows may be available on different machines. They are ordered, for each operation  $i$ , in the increasing order of the  $\alpha_k^i$ ,  $k = 1, 2, \dots, q_i$ . When two  $\alpha_k^i$  are equal, the order is the increasing order of the  $\beta_k^i$ . There are no conflicts between the resources; in other words, the same machine cannot be used to perform different operations for a same product. The last window  $q_i$  associated with each operation  $i$ , is such that  $\beta_{q_i}^i = +\infty$ .

A feasible solution to the problem associated with a set  $[\alpha_{r_i}^i, \beta_{r_i}^i]$  of idle windows (where  $i = 1, 2, \dots, n$  and  $r_i = 1, 2, \dots, q_i$ ) is a set of operation starting times  $\{v_i\}_{i=1, 2, \dots, n}$  which verify, for  $i = 1, 2, \dots, n$ :

$$\alpha_{r_i}^i \leq v_i \quad (1)$$

$$v_i + \theta_i \leq \beta_{r_i}^i, \text{ if } \Gamma_i^+ = \emptyset \quad (2)$$

$$v_j \leq \beta_{r_i}^i, \forall j \in \Gamma_i^+, \text{ if } \Gamma_i^+ \neq \emptyset \quad (3)$$

$$v_i + \theta_i \leq v_j, \forall j \in \Gamma_i^+, \text{ if } \Gamma_i^+ \neq \emptyset \quad (4)$$

$$v_j \leq v_i + \theta_i + \delta_i, \forall j \in \Gamma_i^+, \text{ if } \Gamma_i^+ \neq \emptyset \quad (5)$$

Note that inequalities (4) and (5) allowed to assign to operation  $i$  a processing time  $w_i \in [\theta_i, \theta_i + \delta_i]$  such that  $v_i + w_i = v_j$ ,  $j \in \Gamma_i^+$ . Note also that if  $\Gamma_i^+ = \emptyset$ , then  $w_i = \theta_i$  since the goal consists in minimizing the makespan. A feasible solution is optimal if  $\max_{i \in \Gamma_i^+} (v_i + \theta_i)$  is minimal.

The goal is to schedule the operations required to manufacture a product  $P$  whose assembly process is  $A$ , in order to minimise the makespan, i.e. to complete the product as soon as possible. This scheduling has to be performed as soon as the product arrives in the system. Due to the intensity of the flow of products, it is not allowed to reschedule products which have been previously scheduled.

NB. The scheduling problems with no-wait and blocking are two particular cases of the problem presented in this paper. In the first case (no-wait problem)  $\delta_i = 0$  while  $\delta_i = +\infty$  in the blocking problem.

#### 4. COMPUTATION OF AN OPTIMAL SOLUTION: ALGORITHM 1

The idea behind the algorithm is quite simple and can be explained considering figure 1. Let us assume that the window assigned to each operation is known.

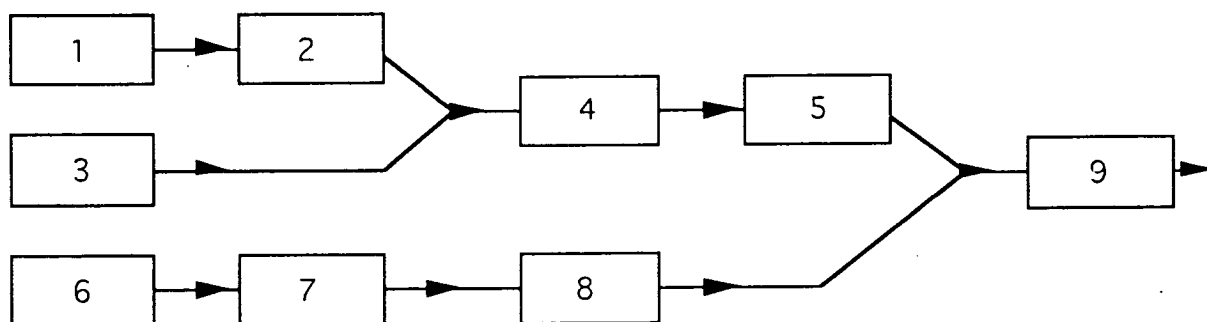


Figure 1. An assembly process composed by 9 operations

In this assembly process, there are two assembly operations: operation 4 and operation 9, the other ones being processing operations. We call "branch" a series of operations which are successors or predecessors from each other, each operation of a branch having at most one predecessor if its predecessor belong to the same branch. The process represented in figure 1 has five branches:  $B_1 = \{1, 2\}$ ,  $B_2 = \{3\}$ ,  $B_3 = \{4, 5\}$ ,  $B_4 = \{6, 7, 8\}$  and  $B_5 = \{9\}$ .  $B_1$  and  $B_2$  are called "predecessor branches" of  $B_3$ .

To schedule this process, we start considering branches which are predecessors of the same branch and which either have no predecessor, or whose predecessors are already been considered. In this case, we first consider  $B_1$  and  $B_2$  and we compute, for each of them, the minimal completion time. The maximal value of these times is the minimal starting time of  $B_3$ . Assume that the window where operation  $i$  should be located is  $[\alpha_i, \beta_i]$ , for  $i \in \{1, 2, \dots, 9\}$ .



We compute successively  $u_i$ , lower bound of the minimal starting time of operation  $i$ , for  $i \in \{1, 2, \dots, 9\}$ :

- operation 1:  $u_1 = \alpha_{r_1}^1$ ,
- operation 2:  $u_2 = \max(\alpha_{r_2}^2, u_1 + \theta_1)$ ,
- operation 3:  $u_3 = \alpha_{r_3}^3$ ,
- operation 4:  $u_4 = \max(\alpha_{r_4}^4, u_2 + \theta_2, u_3 + \theta_3)$ ,
- operation 5:  $u_5 = \max(\alpha_{r_5}^5, u_4 + \theta_4)$ ,
- operation 6:  $u_6 = \alpha_{r_6}^6$ ,
- operation 7:  $u_7 = \max(\alpha_{r_7}^7, u_6 + \theta_6)$ ,
- operation 8:  $u_8 = \max(\alpha_{r_8}^8, u_7 + \theta_7)$ ,
- operation 9:  $u_9 = \max(\alpha_{r_9}^9, u_5 + \theta_5, u_8 + \theta_8)$ .

Thus, the minimal completion time of operation 9 is  $u_9 + \theta_9$ . The optimal makespan if the set of windows under consideration can lead to a feasible solution is  $z = u_9 + \theta_9$ . We then apply a backward process starting from  $u_9$ . The goal of this backward process is to define the starting times of the operations, keeping in mind that the completion time of one operation is the starting time of its successors. So, we obtain successively  $v_i$ , the minimal starting time of operation  $i$ , for  $i \in \{1, 2, \dots, 9\}$ :

- operation 9:  $v_9 = u_9$ ,
- operation 5:  $v_5 = \max(u_5, v_9 - \theta_5 - \delta_5)$ ,
- operation 4:  $v_4 = \max(u_4, v_5 - \theta_4 - \delta_4)$ ,
- operation 2:  $v_2 = \max(u_2, v_4 - \theta_2 - \delta_2)$ ,
- operation 1:  $v_1 = \max(u_1, v_2 - \theta_1 - \delta_1)$ ,
- operation 3:  $v_3 = \max(u_3, v_4 - \theta_3 - \delta_3)$ ,
- operation 8:  $v_8 = \max(u_8, v_9 - \theta_8 - \delta_8)$ ,
- operation 7:  $v_7 = \max(u_7, v_8 - \theta_7 - \delta_7)$ ,
- operation 6:  $v_6 = \max(u_6, v_7 - \theta_6 - \delta_6)$ .

If the operation  $i$  cannot be completed before  $\beta_{r_i}^i$ , we replace window  $[\alpha_{r_i}^i, \beta_{r_i}^i]$  by  $[\alpha_{r_i+1}^i, \beta_{r_i+1}^i]$ , and we restart the process. Better, we can replace the window  $[\alpha_{r_i}^i, \beta_{r_i}^i]$  by the first window  $[\alpha_{r_i+k_i}^i, \beta_{r_i+k_i}^i]$  (i.e.  $k_i$  is as low as possible and  $k_i \geq 1$ ) such that the operation  $i$  can be completed before  $\beta_{r_i+k_i}^i$ .

Otherwise the  $v_i$ ' values are the solution of the problem. This simple example illustrates the general algorithm proposed hereafter.

#### **Algorithm 1 - Assembly**

1. Set  $r_i = 1$ , for  $i \in \{1, 2, \dots, n\}$
2. Forward process
  - 2.1. Set  $E = \{1, 2, \dots, n\}$

- 2.2. While  $E \neq \emptyset$ 
  - 2.2.1. Select  $i \in E$  such that  $\Gamma_i^- \cap E = \emptyset$
  - 2.2.2. If  $\Gamma_i^- = \emptyset$  then set  $u_i = \alpha_{r_i}^i$ , else set  $u_i = \max\left(\alpha_{r_i}^i, \max_{j \in \Gamma_i^-} (u_j + \theta_j)\right)$
  - 2.2.3. Set  $E = E \setminus \{i\}$
- 2.3. Set  $z = \max_{i/\Gamma_i^+ = \emptyset} (u_i + \theta_i)$ ,  $z$  is the optimal makespan if there exists a feasible solution for the selected windows
3. Backward process
  - 3.1. Set  $E = \{1, 2, \dots, n\}$
  - 3.2. While  $E \neq \emptyset$ 
    - 3.2.1. Select  $i \in E$  such that  $\Gamma_i^+ \cap E = \emptyset$
    - 3.2.2. If  $\Gamma_i^+ = \emptyset$  then set  $v_i = u_i$ , else set  $v_i = \max\left(u_i, \max_{j \in \Gamma_i^+} (v_j) - \theta_i - \delta_i\right)$
    - 3.2.3. Set  $E = E \setminus \{i\}$
4. Test
  - 4.1. Set  $E = \{1, 2, \dots, n\}$
  - 4.2. For any  $i \in E$  such that  $\Gamma_i^+ = \emptyset$ ,
    - 4.2.1. While  $v_i + \theta_i > \beta_{r_i}^i$ , set  $r_i = r_i + 1$
  - 4.3. For any  $i \in E$  such that  $\Gamma_i^+ \neq \emptyset$ ,
    - 4.3.1. Set  $j$ , such that  $\{j\} = \Gamma_i^+$
    - 4.3.2. While  $v_i > \beta_{r_i}^i$ , set  $r_i = r_i + 1$
  - 4.4. If none of the  $r_i$  has been modified, then  $z$  is the minimal makespan and  $v_i$  are the earliest starting time of the operations corresponding to  $z$ , else return to step 2.0.

Results 1 and 2 presented hereafter guarantee that the solution obtained by applying this algorithm is optimal.

### Result 1

Algorithm 1 leads to a feasible solution.

### Proof:

**a. We prove that  $\alpha_{r_i}^i \leq v_i$ ,  $\forall i \in \{1, 2, \dots, n\}$ .**

According to step 2.2.2 in algorithm 1,  $\alpha_{r_i}^i \leq u_i$ ,  $\forall i \in \{1, 2, \dots, n\}$ . Furthermore, with considering step 3.2.2,  $u_i \leq v_i$ ,  $\forall i \in \{1, 2, \dots, n\}$ . (6)

Finally:  $\alpha_{r_i}^i \leq v_i$ : this proves inequalities (1).

**b. We prove that  $v_i + \theta_i \leq v_j$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

According to step 3.2.2 in algorithm 1, for any  $i = 1, 2, \dots, n$  such that  $\Gamma_i^+ \neq \emptyset$ ,  $v_i + \theta_i = \max\left(u_i + \theta_i, \max_{j \in \Gamma_i^+} (v_j) - \delta_i\right)$ . Thus,  $v_i + \theta_i \leq \max\left(u_i + \theta_i, \max_{j \in \Gamma_i^+} (v_j)\right)$ . (7)

But according to step 2.2.2, whatever  $j \in \Gamma_i^+$ ,  $u_i + \theta_i \leq u_j$ . (8)

Using inequality (8) in (7), we obtain:  $v_i + \theta_i \leq \max\left(\max_{j \in \Gamma_i^+}(u_j), \max_{j \in \Gamma_i^+}(v_j)\right)$  and using (6),  $v_i + \theta_i \leq v_j$ ,  $\forall j \in \Gamma_i^+$ : this proves inequalities (2).

**c. We prove that  $v_j \leq v_i + \theta_i + \delta_i$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

From step 3.2.2 in algorithm 1, we derive:  $v_i + \theta_i + \delta_i = \max\left(u_i + \theta_i + \delta_i, \max_{j \in \Gamma_i^+}(v_j)\right)$ . As a consequence:  $v_j \leq v_i + \theta_i + \delta_i$ ,  $\forall j \in \Gamma_i^+$ : this proves inequalities (3).

**d. We prove that  $v_i + \theta_i \leq \beta_{r_i}^i$ ,  $\forall i \in \{1, 2, \dots, n\}$  such that  $\Gamma_i^+ = \emptyset$ .**

Since we set  $r_i = r_i + 1$ , when  $v_i + \theta_i > \beta_{r_i}^i$  (see step 4.2.1 in algorithm 1), and since the last window  $q_i$  associated with each operation  $i$ , is such that  $\beta_{q_i}^i = +\infty$ , then the first solution obtained verifies:  $v_i + \theta_i \leq \beta_{r_i}^i$ : this proves inequalities (4).

**e. We prove that  $v_j \leq \beta_{r_i}^i$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

Since we set  $r_i = r_i + 1$ , when  $v_j > \beta_{r_i}^i$  (see step 4.3.2 in algorithm 1), and since the last window  $q_i$  associated with each operation  $i$ , is such that  $\beta_{q_i}^i = +\infty$ , then the first solution obtained verifies:  $v_j \leq \beta_{r_i}^i$ : this proves inequalities (5).

QED

## Result 2

In the first feasible solution obtained by applying algorithm 1, all the completion times, and thus the starting times of the operations are minimal. As a consequence, this solution is optimal.

### Proof:

**a. For a given set of idle windows, we prove that, if it exists a feasible solution, the solution obtained by applying steps 2 and 3 in algorithm 1 is such that no other solution leads to a set of completion times which are lower.**

Considering step 2.2.2 in algorithm 1, and keeping in mind that  $\theta_i$  is the minimal processing time for operation  $i$ , we see that the  $u_i$ ' values cannot be reduced. As a consequence, considering step 3.2.2, and keeping in mind that  $\theta_i + \delta_i$  is the maximal processing time for operations  $i$ , we claim that the  $v_i$ ' values cannot be reduced either.

As a consequence if the solution obtained by applying step 2 and 3 of algorithm 1 to a given set of windows  $\left\{\left[\alpha_{r_i}^i, \beta_{r_i}^i\right]\right\}_{i=1, 2, \dots, n}$  does not verify constraint (2), or (3) for at least one  $i \in \{1, 2, \dots, n\}$ , no feasible solution exists for this set of windows. Furthermore, if the solution obtained is feasible, it is composed with the lower possible starting times of the operations. A consequence of this proof is that the completion times of the operations can not be reduced since they are starting times of the next operation. The ones which do not have a successor are also minimal since their processing time is set at this minimal value.

This completes the first part of the proof.

**b. We prove that the first feasible solution obtained by applying algorithm 1 is optimal.**

If the solution derived from the set of windows having the minimal lower bounds  $\alpha_{r_i}^i$  is feasible, it is optimal, and the starting time of the operations are minimal for this set of windows (see a.). Furthermore, since the completion times increase with the  $\alpha_{r_i}^i$  values, the solution is optimal and the starting time are minimal for any set of idle windows  $\{\{\alpha_{r_i}^i, \beta_{r_i}^i\}\}_{i=1, 2, \dots, n}$ .

Moreover, if the operation  $i$  is such that  $\Gamma_i^+ \neq \emptyset$ ,  $\{j\} = \Gamma_i^+$  and  $v_j > \beta_{r_i}^i$ , then any set of windows containing  $[\alpha_{r_i}^i, \beta_{r_i}^i]$  will lead to the same inequality and thus will not lead to a feasible solution. In the same way, if the operation  $i$  is such that  $\Gamma_i^+ = \emptyset$  and  $v_i + \theta_i > \beta_{r_i}^i$ , then any set of windows containing  $[\alpha_{r_i}^i, \beta_{r_i}^i]$  will not lead to a feasible solution.

This completes the proof.

QED

Note that step 4 in algorithm 1 is executed at most  $q$  times, where  $q = \sum_{i=1}^n q_i$  is the number of idle windows. Thus, the complexity of the algorithm 1 is  $o(n.q)$ .

## 5. REDUCTION OF THE CYCLE TIME: ALGORITHM 2

The optimal solution obtained using algorithm 1 does not lead to a minimal use of the resources. The goal of this section is to show how to proceed to reduce the cycle time and, as a consequence, the global use of the resources.

We propose algorithm 2 and we prove that the cycle time obtained by applying this algorithm is minimal.

### *Algorithm 2 - Reduction of the cycle time*

We order the windows  $[\alpha_{s_i}^i, \beta_{s_i}^i]$ , for each operation  $i$ , in the increasing order of the  $\beta_k^i$ ,  $k = 1, 2, \dots, n_i$ . When two  $\beta_k^i$  are equal, the order is the decreasing order of the  $\alpha_k^i$ . The  $v_i$  values are those computed in algorithm 1.

1. Set  $s_i = n$ , for  $i \in \{1, 2, \dots, n\}$
2. Foreward process
  - 2.1. Set  $E = \{1, 2, \dots, n\}$
  - 2.2. While  $E \neq \emptyset$ 
    - 2.2.1. Select  $i \in E$  such that  $\Gamma_i^- \cap E = \emptyset$
    - 2.2.2. If  $\Gamma_i^- = \emptyset$  then set  $w_i = +\infty$ , else set  $w_i = \min_{k \in \Gamma_i^-} (x_k)$
    - 2.2.3. If  $\Gamma_i^+ = \emptyset$  then set  $x_i = \min(v_i + \theta_i, \beta_{s_i}^i, w_i + \theta_i + \delta_i)$ ,  
else  $x_i = \min(\beta_{s_i}^i, w_i + \theta_i + \delta_i)$
    - 2.2.4. Set  $E = E \setminus \{i\}$
3. Backward process
  - 3.1. Set  $E = \{1, 2, \dots, n\}$

3.2. While  $E \neq \emptyset$

3.2.1. Select  $i \in E$  such that  $\Gamma_i^+ \cap E = \emptyset$

3.2.2. If  $\Gamma_i^+ = \emptyset$  then set  $y_i = \min(x_i - \theta_i, w_i)$ ,

else  $y_i = \min\left(\min_{j \in \Gamma_i^+}(y_j) - \theta_i, x_i - \theta_i, w_i\right)$

3.2.3. Set  $E = E \setminus \{i\}$

4. Test

4.1. Set  $E = \{1, 2, \dots, n\}$

4.2. For any  $i \in E$

4.2.1. While  $y_i < \alpha_{s_i}^i$ , set  $s_i = s_i - 1$

4.3. If none of the  $s_i$  has been modified, then  $y_i$  are the latest starting time of the operations corresponding to  $z$ , else return to step 2.0.

### Result 3

Algorithm 2 transforms the optimal solution obtained by applying algorithm 1 into the optimal one in which the starting time of the operations are maximal. In other words, the solution obtained by applying algorithm 2 is optimal and minimize the use of resources.

### Proof:

**a. We prove that  $\alpha_{s_i}^i \leq y_i$ ,  $\forall i \in \{1, 2, \dots, n\}$ .**

Since we set  $s_i = s_i - 1$ , when  $y_i < \alpha_{s_i}^i$  (see step 4.2.1 in algorithm 2), and since the window  $r_i$  associated with each operation  $i$  in algorithm 1 is such that a feasible solution exists, then the first solution obtained verifies:  $\alpha_{s_i}^i \leq y_i$ : this proves inequalities (1).

**b. We prove that  $y_i + \theta_i \leq \beta_{s_i}^i$ ,  $\forall i \in \{1, 2, \dots, n\}$  such that  $\Gamma_i^+ = \emptyset$ .**

According to step 3.2.2 in algorithm 2, for any  $i = 1, 2, \dots, n$  such that  $\Gamma_i^+ = \emptyset$ ,  $y_i \leq x_i - \theta_i$ . (9)

Furthermore, considering step 2.2.3,  $x_i \leq \beta_{s_i}^i$ .

Finally:  $y_i + \theta_i \leq \beta_{s_i}^i$  which proves inequalities (2).

**c. We prove that  $y_j \leq \beta_{s_i}^i$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

According to step 2.2.3 in algorithm 2,  $x_i \leq \beta_{s_i}^i$ ,  $\forall i \in \{1, 2, \dots, n\}$ .

Furthermore, considering step 2.2.2,  $w_i \leq x_i$ ,  $\forall j \in \Gamma_i^+$ . (10)

According to step 3.2.2,  $y_j \leq w_j$ ,  $\forall j \in \{1, 2, \dots, n\}$ . (11)

Finally:  $y_j \leq \beta_{s_i}^i$ ,  $\forall j \in \Gamma_i^+$ : this proves inequalities (3).

**d. We prove that  $y_i + \theta_i \leq y_j$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

From step 3.2.2 in algorithm 2, we derive:  $y_i + \theta_i \leq \min_{j \in \Gamma_i^+}(y_j)$ . As a consequence:  $y_i + \theta_i \leq y_j$ ,  $\forall j \in \Gamma_i^+$ : this proves inequalities (4).

**e. We prove that  $y_j \leq y_i + \theta_i + \delta_i$ ,  $\forall i \in \{1, 2, \dots, n\} \forall j \in \Gamma_i^+$ .**

According to step 3.2.2 in algorithm 2, for any  $i = 1, 2, \dots, n$  such that  $\Gamma_i^+ \neq \emptyset$ ,  

$$y_i + \theta_i + \delta_i = \min \left( \min_{j \in \Gamma_i^+} (y_j) + \delta_i, x_i + \delta_i, w_i + \theta_i + \delta_i \right).$$

$$\text{Thus, } \min \left( \min_{j \in \Gamma_i^+} (y_j), x_i, w_i + \theta_i + \delta_i \right) \leq y_i + \theta_i + \delta_i. \quad (12)$$

$$\text{But according to step 2.2.3, } x_i \leq w_i + \theta_i + \delta_i. \quad (13)$$

Using inequality (13) in (12), we obtain:  $\min \left( \min_{j \in \Gamma_i^+} (y_j), x_i \right) \leq y_i + \theta_i + \delta_i$  and using relations (10) and (11),  $y_j \leq y_i + \theta_i + \delta_i, \forall j \in \Gamma_i^+$ , since the successor  $j$  is unique: this proves inequalities (5).

At this point, since relations (1) - (5) are satisfied, we can claim that the solution  $\{y_i\}_{i=1, 2, \dots, n}$  is feasible.

**f. We prove that  $y_i \leq v_i, \forall i \in \{1, 2, \dots, n\}$  such that  $\Gamma_i^+ = \emptyset$ .**

According to step 2.2.3 in algorithm 2, for any  $i = 1, 2, \dots, n$  such that  $\Gamma_i^+ = \emptyset$ ,  $x_i - \theta_i \leq v_i$ . But according to inequality (9),  $y_i \leq v_i$ .

We know that the  $v_i$ ' values are minimal and optimal (see result 2). Since the  $y_i$ ' values lead to a feasible solution, and since  $y_i \leq v_i$  for any  $i = 1, 2, \dots, n$  such that  $\Gamma_i^+ = \emptyset$ , the criterion associated with  $\{y_i\}_{i=1, 2, \dots, n}$  is at least as less as the criterion associated with  $\{v_i\}_{i=1, 2, \dots, n}$ . Thus  $\{y_i\}_{i=1, 2, \dots, n}$  is an optimal solution.

**g. For a given set of idle windows, we prove that the solution obtained by applying steps 2 and 3 in algorithm 2 is such that no other solution leads to a set of starting times which are greater.**

Considering step 2.2.3 in algorithm 2, and keeping in mind that  $\theta_i + \delta_i$  is the maximal processing time for operation  $i$ , we see that the  $x_i$ ' values, which are upper bounds of the finishing times, cannot be increased. As a consequence, according to step 2.2.2, we see that the  $w_i$ ' values, which are upper bounds of the starting times, cannot be increased. Thus, considering step 3.2.2, and keeping in mind that  $\theta_i$  is the minimal processing time for operation  $i$ , we claim that the starting times  $y_i$  of operations  $i$  cannot be increased either.

**h. We prove that the first feasible solution obtained by applying algorithm 2 is the optimal one in which the starting time of the operations are maximal.**

If the solution derived from the set of windows having the maximal upper bounds  $\beta_{s_i}^i$  is feasible, it is optimal, and the starting time of the operations are maximal for this set of windows (see g.). Furthermore, since the starting times decrease with the  $\beta_{s_i}^i$ ' values, the solution is optimal and the starting time are maximal for any set of idle windows  $\{[\alpha_{s_i}^i, \beta_{s_i}^i]\}_{i=1, 2, \dots, n}$ .

Moreover, if the operation  $i$  is such that  $y_i < \alpha_{s_i}^i$ , then any set of windows containing  $[\alpha_{s_i}^i, \beta_{s_i}^i]$  will lead to the same inequality and thus will not lead to a feasible solution.

This completes the proof.

QED

Note that step 4 in algorithm 2 is executed at most  $q$  times, where  $q = \sum_{i=1}^n q_i$  is the number of idle windows. Thus, the complexity of the algorithm 2 is  $o(n.q)$ .

## 6. EXAMPLE

The assembly process represented in Figure 1 is composed with 9 operations. Each operation is defined by  $\theta_i$ , its minimal processing time, and  $\theta_i + \delta_i$ , its maximal processing time. Since the system is fully automated, components cannot be stored in front of the resources. In this example, operations (except operations 6 and 7, which are chemical treatments) can be extended as much as necessary, but in this case, machines are unavailable for other operations until the current operation stops.

**Table 1 - Processing times**

i	1	2	3	4	5	6	7	8	n = 9
$\theta_i$	3.	2.	2.	3.	2.	3.	2.	3.	2.
$\delta_i$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0.	0.	$+\infty$	$+\infty$

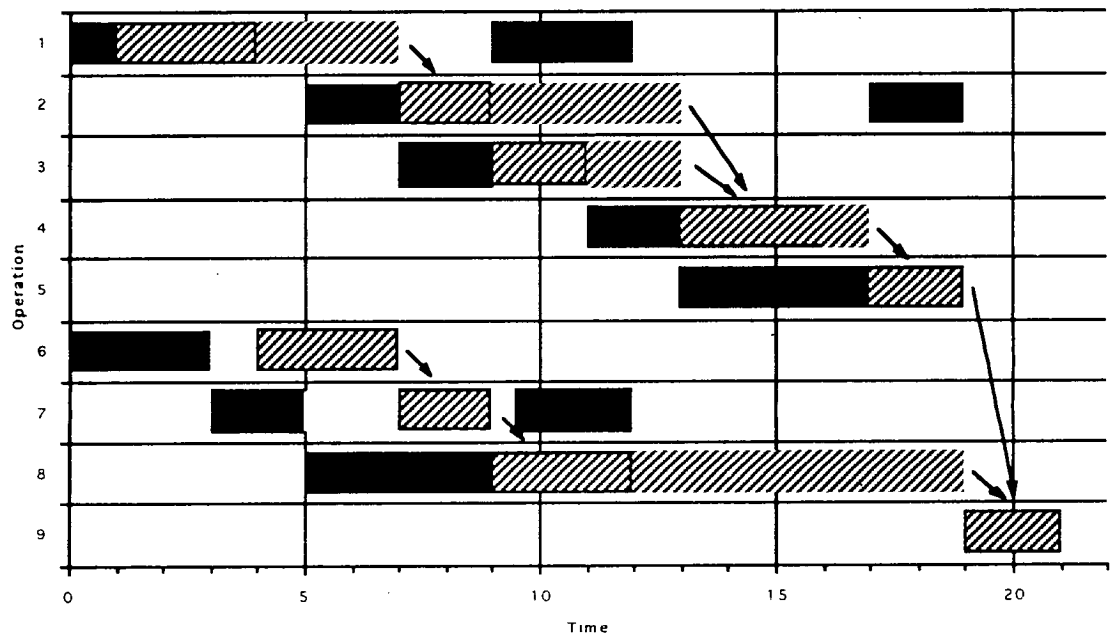
Each operation is assigned to a given machine. Since several operations are previously scheduled, these 9 operations must be performed in one of the idle periods  $[\alpha_r^i, \beta_r^i]$  available on the resource.

**Table 2 - Available windows**

i	1	2	3	4	5	6	7	8	9
$q_i$	2	3	2	2	2	1	3	2	1
$[\alpha_1^i, \beta_1^i]$	[1, 9]	[0, 5]	[0, 7]	[0, 11]	[0, 13]	[3, $\infty$ [	[0, 3]	[0, 5]	[0, $\infty$ [
$[\alpha_2^i, \beta_2^i]$	[12, $\infty$ [	[7, 17]	[9, $\infty$ [	[13, $\infty$ [	[17, $\infty$ [		[5, 9.5]	[9, $\infty$ [	
$[\alpha_3^i, \beta_3^i]$		[19, $\infty$ [					[12, $\infty$ [		

By applying algorithm 1, we obtain the optimal solution in which the starting times of operations are minimal.

**Chart 1 - Minimal starting times**



Busy periods



Operations scheduled by applying algorithm 1  
(minimal processing time)



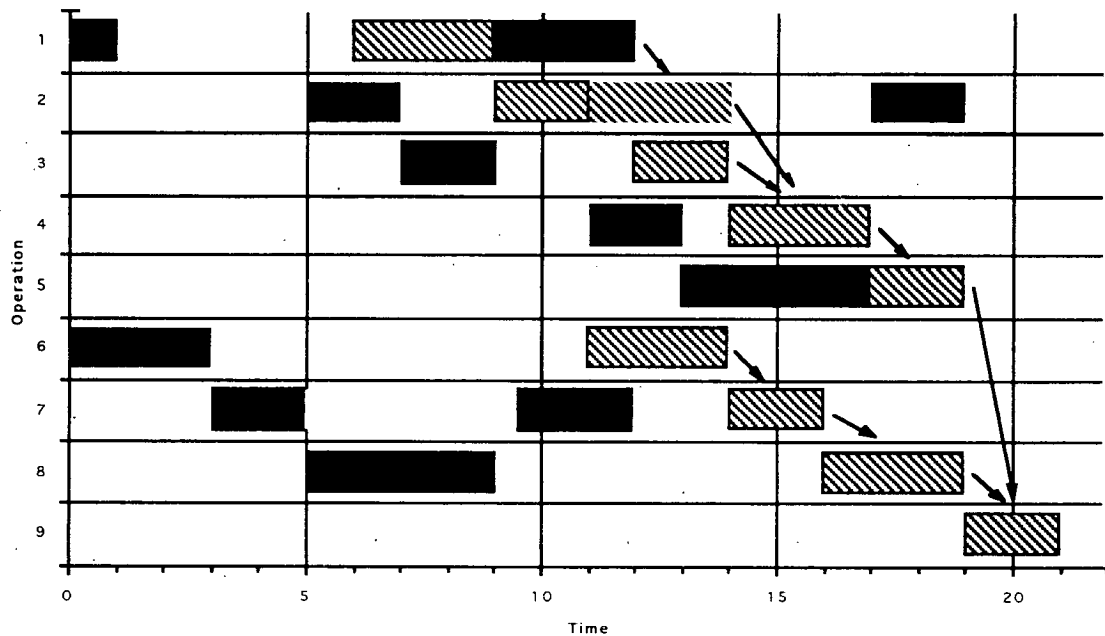
Operations scheduled by applying algorithm 1  
(resulting processing time)








By applying algorithm 2, we obtain the optimal solution in which the starting times of operations are maximal.

**Chart 2 - Maximal starting times**



Busy periods 

Operations scheduled by applying algorithm 2  
(minimal processing time) 

Operations scheduled by applying algorithm 2  
(resulting processing time) 

As we can see in this example, algorithm 2 refine the schedule by pushing to the right, as much as possible, the starting times of the operations. The consequence is an overall reduction of the use of the resources.

## 7. CONCLUSION

The algorithms presented in this paper are real time scheduling algorithms which allow to schedule new assembly process as soon as possible. As a consequence, the first idle periods used are the ones which are the closest to the current time. This guarantees a good use of the resource. Furthermore, by controlling the flexibility of the system (i.e. the  $\delta_i$ ' values), it is possible to reduce the WIP at the expense of the use of resources.

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